

# $\pi\pi$ -scattering lengths at $\mathcal{O}(p^6)$ revisited

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This article completes a former work where part of the  $\mathcal{O}(p^6)$  low-energy constants entering in the  $\pi\pi$  scattering were estimated. Some resonance contributions were missed in former calculations and slight differences appeared with respect to our outcome. Here, we provide the full results for all the contributing  $\mathcal{O}(p^6)$  couplings. We also perform a reanalysis of the hadronic inputs used for the estimation (resonance masses, widths...). Their reliability was checked together with the impact of the input uncertainties on the determinations of the chiral couplings and the scattering lengths  $a_J^I$ . Our outcome is found in agreement with former works though with slightly larger errors. However, the effect in the final values of the  $a_J^I$  is negligible after combining them with the other uncertainties. Based on this consistency, we conclude that the previous scattering length determinations seem to be rather solid and reliable, with the  $\mathcal{O}(p^6)$  low-energy constants quite under control. Nevertheless, the uncertainties found in the present work point out the limitation on further improvements unless the precision of the  $\mathcal{O}(p^6)$  couplings is properly increased.

## I. INTRODUCTION

In a previous work [1, 2], we provided a set of predictions for some of the  $\mathcal{O}(p^6)$  low energy constants (LECs)  $r_i$  related to the  $\pi\pi$ -scattering amplitude [3, 4]. The  $\mathcal{O}(p^6)$  chiral perturbation theory couplings  $r_2, \dots, r_6$  were determined there in the large- $N_C$  limit [5] by means of once-subtracted partial-wave dispersion relations, where they were provided in terms of the ratios of widths over masses,  $\Gamma_R/M_R^3$  and  $\Gamma_R/M_R^5$ . The limit of a large number of colours  $N_C \rightarrow \infty$  [5], is a key ingredient of the study, becoming the strong dynamics greatly simplified and being the dominant contribution provided by the tree-level meson exchanges. At large  $N_C$ , the relevant resonances for  $\pi\pi$ -scattering are the I=1 vector and the  $\bar{u}u + \bar{d}d$  component of the scalar with I=0. We will denote these large- $N_C$  states, respectively, as  $\rho$  and  $\sigma$  all along the paper. We will also consider just the contribution from the lightest multiplets: the single resonance approximation will be assumed.

The new predictions in Ref. [2] found that some scalar resonance contributions to the  $\mathcal{O}(p^6)$  LECs had been actually missed in former estimates [3]. This produced small variations on the  $\pi\pi$ -scattering lengths [2], of the order of the current errors [6]. However, in order to make a thorough analysis, we complete our former study and provide the remaining  $\mathcal{O}(p^6)$  LECs contributing to  $\pi\pi$ -scattering. In the standard  $\mathcal{O}(p^6)$  chiral perturbation theory calculation [3], the scattering lengths  $a_J^I$  depend on  $r_1, \dots, r_6$ . Alternatively, if they are determined through the dispersive method in Ref. [6], one needs to provide  $r_{S_2}$  instead of  $r_5$  and  $r_6$ .

In addition, we have decided to redo the whole analysis of the resonance inputs. This has enabled us with at least a minimal control of the uncertainties in the LECs and the corresponding scattering lengths. This allows a better understanding of what are the relevant parameters that determine the low energy scattering and the main sources of errors. The vector mesons are found to fit very well within a  $U(3)$  large- $N_C$  multiplet and their properties are quite under control. On the other hand, the current knowledge of the lightest large- $N_C$  scalars is rather poor. Thus, our revised analysis of the  $\pi\pi$ -scattering lengths is also motivated by the need of improving the current picture of the lowest lying hadronic resonances [7, 8, 9].

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In Section II, we present the former LEC determinations and the reanalysis that we propose. The phenomenological inputs required for the determination of the LECs are studied in full detail in Section III, keeping a careful control of the possible uncertainties. Based on this, the values of the  $\mathcal{O}(p^6)$  low-energy constants  $r_i$  are estimated in Section IV, leading to a series of new predictions for the  $\pi\pi$  scattering lengths in Section V. The various results and conclusions are gathered in Section VI. Finally, some more technical details are relegated to the Appendices.

## II. LEC ESTIMATES THROUGH RESONANCE SATURATIONS

The current calculations in chiral perturbation theory ( $\chi$ PT) have already reached the two loop order [3, 4]. However, in order to extract quantities such as the scattering lengths one needs to eventually input the  $\mathcal{O}(p^6)$  low-energy constants. Since in many cases it is not possible to determine these couplings directly from the phenomenology, one needs to extract their values through alternative procedures. One of the most usual ones is to consider large- $N_C$  estimates based on phenomenological chiral lagrangians [3, 10, 11]. However, the control that one has on these lagrangians and their large- $N_C$  estimation may be unclear. Thus, the uncertainties of the observables under consideration might be larger than what is actually quoted.

In this work, we propose the comparison of three sets of estimates of the low-energy constants. First we will review the currently used values (set A) [3, 4] and then we will present the new numbers after taking into account the scalar meson contributions that were missing in former works (set B) [2]. However, for sake of consistency, we will also redo the analysis of the hadronical inputs and provide the newly calculated LECs (set C).

### • Set A:

This is the group of estimates commonly employed in nowadays calculations [3, 4]. The  $\chi$ PT couplings are assumed to be determined by the resonance exchanges provided by the phenomenological lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \\ & - \frac{1}{4} \langle \hat{V}_{\mu\nu} \hat{V}^{\mu\nu} \rangle + \frac{1}{2} M_V^2 \langle \hat{V}_\mu \hat{V}^\mu \rangle - \frac{ig_V}{2\sqrt{2}} \langle \hat{V}_{\mu\nu} [u^\mu, u^\nu] \rangle + f_\chi \langle \hat{V}_\mu [u^\mu, \chi_-] \rangle \\ & + \frac{1}{2} \langle \nabla^\mu S \nabla_\mu S \rangle - \frac{1}{2} M_S^2 \langle SS \rangle + c_d \langle S u_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle, \end{aligned} \quad (1)$$

where  $\langle \dots \rangle$  stands for trace in flavour space,  $S$  and  $\hat{V}^\mu$  account respectively for the scalar and vector multiplets. The tensor  $u^\mu$  contains the chiral pseudo-Goldstone and  $\chi_\pm$  is, in addition, proportional to the light quark masses. Their precise definitions can be found in Refs. [3, 4, 10]. Notice that since the lagrangian does not contain resonance mass-splitting terms, the masses  $M_R$  coincide with their chiral limit values, which we denote as  $\overline{M}_R$ . Nevertheless, it can be straightforwardly extended to include the splitting due to the quark masses. This will be studied with full generality in the next sections. From the comparison of the decays  $\rho \rightarrow \pi\pi$ ,  $K^* \rightarrow K\pi$  and other processes, Ref. [3] provided the set of parameters

$$\begin{aligned} M_V &= 770 \text{ MeV}, & g_V &= 0.09, & f_\chi &= -0.03, \\ M_S &= 983 \text{ MeV}, & c_m &= 42 \text{ MeV}, & c_d &= 32 \text{ MeV}. \end{aligned} \quad (2)$$

Taking these inputs and the phenomenological lagrangian (1), they produced the LEC estimates [3],

$$\begin{aligned} r_1^A &= -0.6 \times 10^{-4}, & r_2^A &= 1.3 \times 10^{-4}, & r_3^A &= -1.7 \times 10^{-4}, \\ r_4^A &= -1.0 \times 10^{-4}, & r_5^A &= 1.1 \times 10^{-4}, & r_6^A &= 0.3 \times 10^{-4}, \end{aligned} \quad (3)$$

where the authors already accounted in these numbers the contribution to the  $SU(2)$  LECs coming from the kaon and eta loops [3]:

$$\begin{aligned} r_1^K &= \frac{31F_\pi^2}{5760\pi^2 m_K^2}, & r_2^K &= -\frac{11F_\pi^2}{2304\pi^2 m_K^2}, & r_3^K &= -\frac{29F_\pi^2}{7680\pi^2 m_K^2}, \\ r_4^K &= -\frac{F_\pi^2}{2560\pi^2 m_K^2}, & r_5^K &= \frac{23F_\pi^2}{15360\pi^2 m_K^2}, & r_6^K &= \frac{F_\pi^2}{15360\pi^2 m_K^2}. \end{aligned} \quad (4)$$

These will be also included in our analysis in order to have a clearer comparison with the outcomes obtained in Refs. [3, 6].

In addition to this, the dispersive approach used in Ref. [6] also required the  $\pi\pi$ -scalar form-factor coupling  $r_{S_2}$ , which was estimated to be [4]

$$r_{S_2}^A = -0.3 \times 10^{-4}, \quad (5)$$

with a negligible contribution  $r_{S_2}^K = \frac{F_\pi^2}{1152\pi^2 m_K^2} = 0.03 \cdot 10^{-4}$  from kaon and eta loops [4].

• **Set B:**

However, the resonance estimate from Refs. [3, 6] was incomplete and some relevant contributions were actually missing in their calculation. The presence of the operator  $c_m \langle S\chi_+ \rangle$  in the scalar meson lagrangian produces a tadpole term proportional to the quark mass [12, 13]. Although its effect vanishes in the chiral limit, its contribution becomes relevant at subleading chiral orders.

The  $\mathcal{O}(p^6)$  LECs are recalculated here in full detail, keeping all the possible contribution. No term of the corresponding chiral order under study is neglected. By means of the partial-wave dispersion relations proposed in Refs. [1, 2], most of the  $\mathcal{O}(p^6)$  LECs in the  $\pi\pi$ -scattering ( $r_2, \dots, r_6$ ) are now fixed in terms of the ratios,

$$\begin{aligned} \frac{\Gamma_R}{M_R^3} &= \frac{\bar{\Gamma}_R}{\bar{M}_R^3} \left[ 1 + \alpha_R \frac{m_\pi^2}{\bar{M}_R^2} + \gamma_R \frac{m_\pi^4}{\bar{M}_R^4} + \mathcal{O}(m_\pi^6) \right], \\ \frac{\Gamma_R}{M_R^5} &= \frac{\bar{\Gamma}_R}{\bar{M}_R^5} \left[ 1 + \beta_R \frac{m_\pi^2}{\bar{M}_R^2} + \mathcal{O}(m_\pi^4) \right], \end{aligned} \quad (6)$$

where  $\bar{M}_R$  and  $\bar{\Gamma}_R$  stand for the chiral limit of  $M_R$  and  $\Gamma_R$ , respectively. The constants  $\alpha_R, \beta_R, \gamma_R$  are quark mass independent and rule the  $m_\pi$  corrections in the ratios. The formal expressions for  $r_2, \dots, r_6$  [2] are provided later in Sections IV A–IV C. The resonance lagrangian (1) allows us then to compute the resonance masses and widths at large  $N_C$  in terms of the resonance couplings. Thus, the LEC predictions can be utterly rewritten in terms of the latter. Using exactly the same inputs (2) of set A, one obtains then slightly different values

$$r_2^B = 18 \times 10^{-4}, \quad r_3^B = 0.9 \times 10^{-4}, \quad r_4^B = -1.9 \times 10^{-4}, \quad (7)$$

remaining  $r_5$  and  $r_6$  unchanged, i.e.,  $r_5^B = r_5^A$  and  $r_6^B = r_6^A$  as in Eq. (3). Although the variation in  $r_3, \dots, r_6$  with respect to Set A is relatively mild,  $r_2$  changes by an order of magnitude. This points out the need of a more detailed investigation of the stability of the estimation under modifications in the inputs.

To complete the calculation we needed to extract  $r_1$  (for the direct  $\chi$ PT calculation [3]) and  $r_{S_2}$  (for the dispersive method [6]). These two couplings were out of the reach of the partial-wave dispersion relations proposed in our former works [1, 2]. Here we have used the lagrangian (1). We have performed the explicit field theory calculation of the  $\pi\pi$  scattering  $A(s, t, u)$  and the scalar form factor  $\mathcal{F}_S(s)$ . The details of the calculation are shown in Appendix A.

The corrected determinations derived from the phenomenological lagrangian (1) result

$$r_1 = -\frac{4g_V^2 F^2}{M_V^2} \left( 2 + \frac{4\sqrt{2}f_\chi}{g_V} \right)^2 - \frac{16c_d c_m (8c_d^2 - 17c_d c_m + 12c_m^2)}{M_S^4}, \quad (8)$$

$$r_{S_2} = \frac{8c_m(c_m - c_d)F^2}{M_S^4} - \frac{32c_d^2 c_m^2}{M_S^4}, \quad (9)$$

which for the set A inputs in Eq. (2) yield

$$r_1^B = -2.1 \times 10^{-4}, \quad r_{S_2}^B = -0.3 \times 10^{-4}. \quad (10)$$

Our prediction  $r_{S_2}^B$  agrees accidentally the numerical value  $r_{S_2}^A$  reported in Ref. [4] and the variation in  $r_1$  is also found to be small.

• **Set C:**

The analysis can be further refined. In addition to performing the full computation of the LECs (without neglecting any contribution), a more careful examination of the phenomenological inputs also seems convenient. A more general description of the mesonic interactions is required. For instance, the lagrangian considered in Eq. (1) does not take into account the fact that in physical QCD there is a mass splitting within the  $U(3)$  resonance multiplet. This quark mass effect utterly contributes to the scattering lengths at the same order as the terms already included in the determinations from sets A and B. A similar thing happens with the resonance widths, which accept a more general quark mass splitting pattern.

Actually, it is possible to describe most of the  $U(3)$  breaking without relying in any particular resonance lagrangian realization. Nevertheless, without loss of generality, we will study the vector resonances in both the Proca and antisymmetric formalisms [14]. Although the two representations are physically equivalent, small discrepancies may appear when considering  $m_\pi^2$  corrections only up to a given order. The slight difference that appears when rearranging the experimental information from one formalism to the other will serve us as an estimate of the residual error due to higher order  $m_\pi^2$  contributions. We will also perform a general analysis of the splitting in the resonance masses and widths. This will allow us to fix their dominant chiral corrections, being reflected in a more accurate estimate of the LECs  $r_3, \dots, r_6$ . However, it will be impossible for us to have a full control of the subdominant chiral corrections –that are going to affect  $r_1, r_2$  and  $r_{S_2-}$ . Therefore, we will have to utterly rely on estimates of these LECs based on phenomenological resonance lagrangians like that in Eq. (1). This Set C reanalysis of the resonance parameters will be shown in detail in the next Section.

### III. PHENOMENOLOGY OF THE RESONANCE PARAMETERS

#### A. Mass splitting up to $\mathcal{O}(m_P^2)$

In the large- $N_C$  limit, the mass splitting of the resonance multiplets can be described at leading order by a single operator  $e_m^R$  [15]

$$-\frac{\overline{M}_R^2}{2} \langle RR \rangle + e_m^R \langle RR\chi_+ \rangle, \quad (11)$$

which leads at large  $N_C$  to the mass eigenstates

$$\begin{aligned} M_{I=1}^2 &= \overline{M}_R^2 - 4e_m^R m_\pi^2 + \mathcal{O}(m_P^4), \\ M_{I=\frac{1}{2}}^2 &= \overline{M}_R^2 - 4e_m^R m_K^2 + \mathcal{O}(m_P^4), \\ M_{I=0}^{(\bar{s}s)}^2 &= \overline{M}_R^2 - 4e_m^R (2m_K^2 - m_\pi^2) + \mathcal{O}(m_P^4). \end{aligned} \quad (12)$$

and  $M_{I=0}^{(\bar{u}u+\bar{d}d)} = M_{I=1}$ .

In the case of the vector resonance multiplet,  $e_m^V$  can be fixed through the physical  $\rho(770)$  and  $K^*(892)$  masses, respectively  $M_\rho = 775.5 \pm 0.3$  MeV and  $M_{K^*} = 893.6 \pm 1.9$  MeV [9]. This provides the estimates

$$\overline{M}_V = 764.9 \pm 0.5 \text{ MeV}, \quad e_m^V = -0.217 \pm 0.004. \quad (13)$$

These values and Eq. (12) lead to a prediction for the remaining two states,  $M_{I=0}^{(\bar{s}s)} = 998$  MeV and  $M_{I=0}^{(\bar{u}u+\bar{d}d)} = M_\rho = 776$  MeV. This is in pretty good agreement with the  $\phi(1020)$  and  $\omega(782)$  masses, respectively  $M_\phi = 1019$  MeV and  $M_\omega = 783$  MeV. Hence, alternatively, one can think of extracting the chiral limit resonance mass and the splitting term  $e_m^V$  by means of the  $\rho(770)$  and  $\phi(1020)$  masses, with  $M_\phi = 1019.455 \pm 0.020$  MeV [9]:

$$\overline{M}_V = 763.7 \pm 0.5 \text{ MeV}, \quad e_m^V = -0.2414 \pm 0.0022, \quad (14)$$

with the prediction  $M_{I=1/2} = 906$  MeV, in relatively good agreement with the experimental  $K^*(892)$  mass [9].

The physical masses are indeed slightly different to their large- $N_C$  values. Thus, the relations in Eq. (12) are not exactly fulfilled for the experimental inputs as they gain subleading contributions in  $1/N_C$ . The combination of Eqs. (13) and (14) provide our most reliable estimate of the large- $N_C$  parameters. Considering an interval that covers both determination, we will take as inputs from now on the values

$$\overline{M}_V = 764.3 \pm 1.1 \text{ MeV}, \quad e_m^V = -0.228 \pm 0.015, \quad (15)$$

whose errors exceed those that come from purely the experimental uncertainties in Eqs. (13) and (14).

In the case of the scalars, it is relatively straightforward to identify the lightest  $I = 1$  resonance with the  $a_0(980)$ :  $M_{I=1} = 984.7 \pm 1.2$  MeV [9]. The remaining states of the multiplet are more cumbersome to pin down. The scalar spectrum provided in the large- $N_C$  study of Ref. [15] leaves the light and broad scalar resonances  $\sigma$  and  $\kappa$  out of any classification multiplet. In order to avoid the problem of the mixing of iso-singlet scalars, the analysis is performed with the  $I = 1/2$  state, whose mass in the large- $N_C$  limit is also poorly known. The broad  $\kappa(800)$  seems to be a possible candidate although the first clear  $I = 1/2$  scalar resonance signal is provided by the  $K_0^*(1430)$  [9]. Hence, we take the conservative estimate  $M_{I=1/2} = 1050 \pm 400$  MeV, which ranges from the  $\kappa$  up to the  $K_0^*(1430)$  mass. This leads to the predictions

$$\overline{M}_S = 980 \pm 40 \text{ MeV}, \quad e_m^S = -0.1 \pm 0.9. \quad (16)$$

### B. The splitting of the vector resonance decay width up to $\mathcal{O}(m_P^2)$

The calculation of the quark mass corrections to the width is slightly more complicate and one needs to make some assumption on the structure of the interaction.

If we use the phenomenological lagrangian (1), the vector decay width into two light pseudo-scalars,  $V \rightarrow \phi_1 \phi_2$ , shows the structure

$$\Gamma_{V \rightarrow \phi_1 \phi_2} = C_{V12} \times \frac{M_V^n \rho_{V12}^3}{48 \pi F_1^2 F_2^2} \lambda_{V\pi\pi}^2 \left[ 1 + \epsilon_V \frac{m_1^2 + m_2^2}{2\overline{M}_V^2} + \mathcal{O}(m_P^4) \right]^2, \quad (17)$$

with the phase-space factor

$$\rho_{V12} = \sqrt{\left(1 - \frac{(m_1 + m_2)^2}{M_V^2}\right) \left(1 - \frac{(m_1 - m_2)^2}{M_V^2}\right)}. \quad (18)$$

The  $F_i$  are the physical decay constants of the  $\phi_i$  pseudo-Goldstones ( $F_\pi \simeq 92.4$  MeV and  $F_K \simeq 113$  MeV) and they appear due to the large- $N_C$  wave function renormalization of the light pseudo-scalars [2, 13].  $M_V$  and  $m_i$  corresponds, respectively, to the physical vector and pseudo-scalar masses.  $\lambda_{V\pi\pi}$  is the  $V\phi_1\phi_2$  coupling that rules the decay amplitude in the chiral limit (e.g.  $g_V$  in Eq. (1) [3, 14]) whereas  $\epsilon_V$  rules the quark mass corrections to the amplitude (provided for instance by the coupling  $f_\chi$  in Eq. (1) [3]).  $C_{V12}$  is the corresponding Clebsch-Gordan:  $C_{\rho\pi\pi} = 1$ ,  $C_{K^*K\pi} = 3/4$  and  $C_{\phi K \overline{K}} = 1$ . The vector mass scaling  $n$  is an integer that depends on the resonance lagrangian under consideration:  $n = 5$  in the Proca formalism [3, 14] and  $n = 3$  in the antisymmetric tensor representation [10, 11]. Finally, one power of  $\rho_{V12}$  comes from phase-space and the other two powers are due to the  $J = 1$  spin polarization sum of the squared modulus of the amplitude,  $\sum_\epsilon |\mathcal{M}_{V \rightarrow 12}|^2 = \sum_\epsilon |2\epsilon_\mu p_1^\mu \widetilde{\mathcal{M}}|^2 = M_V^2 \rho_{V12}^2 |\widetilde{\mathcal{M}}|^2$ , where here the  $\epsilon_\mu$  denote the vector meson polarizations.

Thus, in the Proca field realization from Eq. (1) one has [3, 4, 14]

$$n = 5, \quad \lambda_{V\pi\pi} = g_V, \quad \epsilon_V = 4\sqrt{2}f_\chi/g_V. \quad (19)$$

On the other hand, in the antisymmetric tensor representation employed in Resonance Chiral Theory (R $\chi$ T) [10, 11] one has  $\lambda_{V\pi\pi} = G_V$ ,  $n = 3$  and the quark mass corrections  $\epsilon_V$  provided by a combination of some extra  $V\phi\phi$  operators,  $i\lambda_8^V \langle V_{\mu\nu} \{\chi_+, u^\mu u^\nu\} \rangle$ ,  $i\lambda_9^V \langle V_{\mu\nu} u^\mu \chi_+ u^\nu \rangle$  [11]. The detailed discussion on the quark mass splitting operators in the antisymmetric formalism is relegated to Appendix C. Abusing of the notation, we will denote  $f_\chi$  as the effective combination of resonance chiral theory couplings such that for the antisymmetric formalism ( $n = 3$ ) we still keep  $\epsilon_V = 4\sqrt{2}f_\chi/\lambda_{V\pi\pi} = 4\sqrt{2}f_\chi/G_V$ .

In order to remain as general as possible, both vector formalism scalings,  $n = 5$  and  $n = 3$ , are discussed. The following inputs will be used all throughout this paper to estimate the uncertainties of our results [9]:

$$\begin{aligned} F_\pi &= 92.4 \pm 0.3 \text{ MeV}, & F_K &= 113.0 \pm 1.0 \text{ MeV}, \\ m_\pi &= 137.3 \pm 2.3 \text{ MeV}, & m_K &= 495.6 \pm 2.0 \text{ MeV}, \\ \Gamma_{\rho \rightarrow \pi\pi} &= 149.4 \pm 1.0 \text{ MeV}, & \Gamma_{K^* \rightarrow \pi K} &= 50.6 \pm 0.9 \text{ MeV}, \end{aligned} \quad (20)$$

together with the mass parameters  $\overline{M}_V$  and  $e_m^V$  derived in Eq. (15).

The combination of the experimental  $K^*$  and  $\rho$  widths allows us to fix the parameters  $\lambda_{V\pi\pi}$  and  $\epsilon_V$ :<sup>1</sup>

$$\lambda_{V\pi\pi}^{n=5} = g_V = 0.0846 \pm 0.0008, \quad \epsilon_V^{n=5} = 0.01 \pm 0.09, \quad \text{for } n=5, \quad (21)$$

$$\lambda_{V\pi\pi}^{n=3} = G_V = 63.9 \pm 0.6 \text{ MeV}, \quad \epsilon_V^{n=3} = 0.82 \pm 0.10, \quad \text{for } n=3. \quad (22)$$

For  $n = 5$ , this corresponds to the coupling  $f_\chi = (0.2 \pm 1.3) \times 10^{-3}$ , to be compared to the determination  $f_\chi = -0.03$  from Ref. [3]<sup>2</sup>. In the case of the antisymmetric formalism, one obtains  $f_\chi^{n=3} = 9.3 \pm 1.1 \text{ MeV}$ .

The only difference between the Proca ( $n=5$ ) and the antisymmetric tensor formalism ( $n=3$ ) is that in the latter the  $\rho \rightarrow \pi\pi$  width carries the factor  $G_V^2 \left[1 + \epsilon_V^{n=3} \frac{m_\pi^2}{M_V^2}\right]$ . Instead, in the Proca case, this is replaced by  $g_V^2 M_\rho^2 \left[1 + \epsilon_V^{n=5} \frac{m_\pi^2}{M_V^2}\right] \simeq g_V^2 \overline{M}_V^2 \left[1 + (\epsilon_V^{n=5} - 4e_m^V) \frac{m_\pi^2}{M_V^2}\right]$ . Up to higher chiral order corrections, one finds a pretty good agreement between both of them, getting identical results for  $G_V = 63.9 \text{ MeV}$  and  $g_V \overline{M}_V = 64.7 \text{ MeV}$ , and for  $\epsilon_V^{n=3} = 0.8$  and  $(\epsilon_V^{n=5} - 4e_m^V) = 0.9$ .

The obtained values of  $\lambda_{V\pi\pi}$  and  $\epsilon_V$  lead us to a prediction for the  $\phi \rightarrow K\overline{K}$  decay width:

$$\Gamma_{\phi \rightarrow K\overline{K}} = 4.1 \pm 0.8 \text{ MeV}, \quad (23)$$

for both Proca and antisymmetric formalisms. This result is perfectly consistent with the experimental value  $\Gamma_{\phi \rightarrow K\overline{K}}^{\text{Exp}} = 3.54 \pm 0.10 \text{ MeV}$  [9]. At large- $N_C$ , the  $\phi(1020)$  is identified with the  $\bar{s}s$  component of the  $I = 0$  vector.

Likewise, the determination of the coupling  $\lambda_{V\pi\pi}$  produces for the  $\rho \rightarrow \pi\pi$  decay width the chiral limit prediction,

$$\overline{\Gamma}_\rho = \frac{\lambda_{V\pi\pi}^2 \overline{M}_V^n}{48\pi F^4} \longrightarrow \quad \overline{\Gamma}_\rho = 181.9 \pm 3.0 \text{ MeV}, \quad (n=5), \quad (24)$$

$$\overline{\Gamma}_\rho = 177.8 \pm 2.5 \text{ MeV}, \quad (n=3), \quad (25)$$

where we have used the value of  $F = 90.8 \pm 0.3 \text{ MeV}$  derived in Sec. III D.

### C. The decay width for the scalar resonance

In the case of scalar resonance decays, the constraints to the structure of the width are given just by an overall phase-space factor  $\rho_{V12}$  and the  $F_i^{-1}$  factors due to the  $\phi_i$  wave-function renormalizations of the pseudo-Goldstones [13].

The issue that arises here is the current poor knowledge on the structure of the scalar sector in the large- $N_C$  limit. One needs then to be assisted by some phenomenological lagrangian. In the present work, we will assume that our scalar interactions are ruled by the action from Eq. (1) together with the mass splitting operators  $e_m^S$  from Eq. (11). Although many works have tried to pin down the values of the two scalar couplings,  $c_d$  and  $c_m$ , in Eq. (1) [10, 16, 17, 18, 19, 20], still there are no widely accepted results.

The  $a_0 \rightarrow \pi^0\eta$  decay seems to be the only reliable process to estimate  $c_d$  and  $c_m$ . However, this only fixes a combination of the two and one needs to add extra theoretical information. Thus, the study of the high-energy behaviour of the  $K\pi$ ,  $K\eta$ ,  $K\eta'$  scalar form-factors performed in Ref [16] provided the constraints  $c_m = c_d$  and  $4c_d c_m = F^2$ . However, the first constraint seems *a priori* less reliable, as the analysis did not include all the possible quark mass operators contributing at that order (these extra terms in the lagrangian were derived later in Ref. [11]). On the other hand, the relation  $4c_d c_m = F^2$  is more solid, as it stems from the high-energy analysis of the scalar form-factor in the chiral limit.

Hence, the constraint  $4c_d c_m = F^2$  and the  $a_0 \rightarrow \pi\eta$  decay width are used to fix the values of  $c_d$  and  $c_m$ . Thus, the lagrangian (1) yields

$$\Gamma_{a_0 \rightarrow \pi\eta} = \left( \frac{F_1 \cos \theta_1 - \sqrt{2} F_8 \sin \theta_8}{F_1 F_8 \cos(\theta_1 - \theta_8)} \right)^2 \times \frac{c_d^2 M_{a_0}^3 \rho_{a_0\pi\eta}}{24\pi F_\pi^2} \left( 1 - \frac{m_\pi^2 + m_\eta^2}{M_{a_0}^2} + (\epsilon_S + 2) \frac{m_\pi^2}{M_{a_0}^2} \right)^2, \quad (26)$$

<sup>1</sup> Indeed, the system of quadratic equations has two possible solutions for  $\lambda$  and  $\epsilon_V$ . In one case  $\epsilon_V$  is much larger ( $\epsilon_V \simeq -10$ ) than in the other. Since this corresponds to huge chiral corrections, we will only keep the small  $\epsilon_V$  solution ( $|\epsilon_V| \lesssim 1$ ).

<sup>2</sup> If, alternatively, we consider the same inputs as Ref. [3] together with  $F_K = 110 \text{ MeV}$ , then one gets  $f_\chi/g_V \simeq -0.03$ , leading to  $f_\chi \simeq -0.003$ .



where the lagrangian from Eq. (1) yields  $\epsilon_S + 2 = 2c_m/c_d$ . The  $\eta$ - $\eta'$  mixing is given in the basis of the octet  $\eta_8$  and singlet  $\eta_1$  by [21, 22, 23]:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{F} \begin{pmatrix} F_8 \cos \theta_8 & -F_1 \sin \theta_1 \\ F_8 \sin \theta_8 & F_1 \cos \theta_1 \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix}. \quad (27)$$

We consider the inputs  $F_1 = (1.1 \pm 0.1) F_\pi$ ,  $\theta_1 = (-5 \pm 1)^\circ$ ,  $F_8 = (1.3 \pm 0.1) F_\pi$ ,  $\theta_8 = (-20 \pm 2)^\circ$  [21, 22, 23],  $\Gamma_{a_0 \rightarrow \pi\eta} = 75 \pm 25$  MeV,  $m_\eta = 547.9$  MeV [9], together with the inputs considered in previous sections. Most of the error in the next determinations comes from our poor knowledge on the  $a_0(980)$  decay width.

Relying on the estimate of the  $a_0 \rightarrow \eta\pi^0$  decay width from the phenomenological scalar lagrangian (1) [3, 10] and the theoretical constraint  $4c_d c_m = F^2$  [16, 17, 24], one gets the values

$$c_d = 26 \pm 7 \text{ MeV}, \quad c_m = 80 \pm 21 \text{ MeV}. \quad (28)$$

As the error is dominated by the large  $a_0(980)$  width uncertainty, we find that the determination of  $c_d$  and  $c_m$  is completely unaffected by whether one considers the constraint  $4c_d c_m = F^2$  or the approximation  $4c_d c_m = F_\pi^2$ . The chiral limit  $F$  of the pion decay constant  $F_\pi$  is computed in the next section. Indeed, the determination of  $c_d$  is rather model independent since the  $(\epsilon_S + 2)$  term in Eq. (26) has little impact in the  $a_0$  decay width. Thus, if the  $(\epsilon_S + 2)$  term is neglected one gets  $c_d = 31$  MeV.

Taking this into account, the  $\sigma \rightarrow \pi\pi$  decay width is given at large  $N_C$  by

$$\Gamma_\sigma = \frac{3\rho_{\sigma\pi\pi} c_d^2 M_\sigma^3}{16\pi F_\pi^4} \left( 1 + \epsilon_S \frac{m_\pi^2}{M_S^2} \right)^2, \quad (29)$$

where by  $\sigma$  we denote the iso-singlet ( $\bar{u}u + \bar{d}d$ ) scalar, without strange quark content. The quark mass correction  $\epsilon_S$  and the chiral limit of the width are then provided by

$$\epsilon_S = 2 \left( \frac{c_m}{c_d} - 1 \right) = 4 \pm 3, \quad \bar{\Gamma}_\sigma = \frac{3c_d^2 \bar{M}_S^3}{16\pi F^4} = 600 \pm 300 \text{ MeV}. \quad (30)$$

#### D. Chiral corrections to $F_\pi$

In order to extract the LECs related with  $m_\pi^2$  corrections, we will need to know the quark mass dependence of  $F_\pi$ , which can be parametrized in the general form

$$F_\pi = F \left[ 1 + \delta F_{(2)} \frac{m_\pi^2}{M_S^2} + \delta F_{(4)} \frac{m_\pi^4}{M_S^4} + \mathcal{O}(m_\pi^6) \right]. \quad (31)$$

The pion decay constant  $F_\pi$  appears in the calculation when one takes into account the pion wave-function renormalization  $Z_\pi$  that occurs at large  $N_C$ , which obeys  $F_\pi = F \cdot Z_\pi^{-1/2}$  [2, 13].

The scalar lagrangian in Eq. (1) [10] and the mass splitting from Eq.(12) yield

$$\delta F_{(2)} = \frac{4c_d c_m}{F^2}, \quad \delta F_{(4)} = \frac{8c_d c_m}{F^2} \left( \frac{3c_d c_m}{F^2} - \frac{4c_m^2}{F^2} \right) + \frac{16c_d c_m e_m^S}{F^2}. \quad (32)$$

Although  $\delta F_{(2)}$  has the most general structure, this is not true for  $\delta F_{(4)}$ . If a more general set of scalar operators  $\lambda^S \mathcal{O}_S$ ,  $\lambda^{SS} \mathcal{O}_{SS}$ ,  $\lambda^{SSS} \mathcal{O}_{SSS}$  were allowed in the lagrangian [11],  $\delta F_{(4)}$  would gain a whole series of new contributions.

Substituting our former determinations of  $c_d$ ,  $c_m = F^2/4c_d$  and  $e_m^S$  in Eq. (32), one gets

$$\delta F_{(2)} = 1, \quad \delta F_{(4)} = -5 \pm 5, \quad (33)$$

where the large uncertainty comes both from  $e_m^S$  and the 25% error in  $c_d$ . For  $\delta F_{(4)}$  one can use indistinctly  $c_m = F^2/4c_d$  or  $c_m = F_\pi^2/4c_d$ , as the difference results negligible and goes to the next order in the  $m_\pi^2$  expansion. By means of Eqs. (31) and (33) it is then possible to recover the large- $N_C$  value for the pion decay constant in the chiral limit,  $F = 90.8 \pm 0.3$  MeV, in agreement with former large- $N_C$   $\chi$ PT determination [21].

### E. NLO chiral symmetry breaking parameters $\alpha_R, \beta_R$

Combining the information obtained from the mass and width splittings, one can now extract the corresponding quark mass corrections in the ratios  $\Gamma_R/M_R^3$  and  $\Gamma_R/M_R^5$  defined in Eq. (6):

$$\begin{aligned}\alpha_V &= 2\epsilon_V - 2(n-3)e_m^V - 4\delta F_{(2)} \frac{\overline{M}_V^2}{\overline{M}_S^2} - 6, \\ \beta_V &= 2\epsilon_V - 2(n-5)e_m^V - 4\delta F_{(2)} \frac{\overline{M}_V^2}{\overline{M}_S^2} - 6, \\ \alpha_S &= 2\epsilon_S - 4\delta F_{(2)} - 2, \\ \beta_S &= 2\epsilon_S + 4e_m^S - 4\delta F_{(2)} - 2.\end{aligned}\tag{34}$$

Substituting the same experimental inputs as before one obtains

$$\begin{aligned}\alpha_V &= -7.5 \pm 0.3, & \beta_V &= -8.4 \pm 0.3, & \text{for } n=5, \\ \alpha_V &= -6.8 \pm 0.3, & \beta_V &= -7.7 \pm 0.3, & \text{for } n=3.\end{aligned}\tag{35}$$

and for the scalar

$$\alpha_S = 2 \pm 7, \quad \beta_S = 2 \pm 8.\tag{36}$$

Although  $\alpha_R$  is not needed in the present  $\mathcal{O}(p^6)$  LEC study, it is provided here for sake of completeness.

### F. NNLO chiral correction $\gamma_R$

The parameter  $\gamma_R$  -appearing in  $\Gamma_R/M_R^3$  at  $\mathcal{O}(m_\pi^4)$ - is even more complicate to determine than  $\alpha_R$  and  $\beta_R$ .

At next-to-next-to-leading order (NNLO), the resonance masses may suffer corrections due to the chiral operators

$$\frac{\tilde{e}_{m,1}^R}{2\overline{M}_R^2} \langle RR\chi_+\chi_+ \rangle + \frac{\tilde{e}_{m,2}^R}{2\overline{M}_R^2} \langle R\chi_+R\chi_+ \rangle,\tag{37}$$

which combined with the leading and NLO operators in Eq. (11) yield the pattern

$$\begin{aligned}M_{I=1}^2 &= \overline{M}_R^2 - 4e_m^R m_\pi^2 - 4(\tilde{e}_{m,1}^R + \tilde{e}_{m,2}^R) \frac{m_\pi^4}{\overline{M}_R^2}, \\ M_{I=1/2}^2 &= \overline{M}_R^2 - 4e_m^R m_K^2 - \frac{4}{\overline{M}_R^2} [\tilde{e}_{m,1}^R (2m_K^4 - 2m_K^2 m_\pi^2 + m_\pi^4) + \tilde{e}_{m,2}^R m_\pi^4], \\ M_{I=0}^{(\bar{s}s)}^2 &= \overline{M}_R^2 - 4e_m^R (2m_K^2 - m_\pi^2) - 4(\tilde{e}_{m,1}^R + \tilde{e}_{m,2}^R) \frac{(2m_K^2 - m_\pi^2)^2}{\overline{M}_R^2},\end{aligned}\tag{38}$$

and  $M_{I=0}^{(\bar{u}u+\bar{d}d)} = M_{I=1}$ . Actually, the NNLO multiplet splitting will be only relevant for the analysis of the vector parameter  $\gamma_V$  in the Proca formalism ( $n=5$ ). In the remaining cases,  $\tilde{e}_m^R \equiv \tilde{e}_{m,1}^R + \tilde{e}_{m,2}^R$  will not appear. Since these NNLO quark mass corrections may enter in serious competition with those NLO in  $1/N_C$ , an analysis of the vector spectrum in order to fix  $\tilde{e}_m^V$  seems unreliable. Thus, we consider just some conservative bounds. It is not really possible to obtain a tight constraint from the  $I=1$  or  $I=\frac{1}{2}$ . The most stringent bound comes from the  $I=0$  ( $\bar{s}s$ ) state:

$$|\tilde{e}_m^V| \leq \frac{\overline{M}_V^2}{2m_K^2 - m_\pi^2} |e_m^V| \simeq 0.3,\tag{39}$$

where we have demanded that the NNLO could not overcome the NLO contribution to the  $\phi(1020)$  mass (in the quark mass expansion).



Likewise, the expansion of the  $\rho$  and  $\sigma$  resonance widths up to NNLO in  $m_\pi^2$  is given by the parameters  $\tilde{\epsilon}_R$ :

$$\begin{aligned}\Gamma_{\rho \rightarrow \pi\pi} &= \frac{M_\rho^n \rho_{\rho\pi\pi}^3}{48\pi F_\pi^4} \lambda_{V\pi\pi}^2 \left[ 1 + \epsilon_V \frac{m_\pi^2}{\overline{M}_V^2} + \tilde{\epsilon}_V \frac{m_\pi^4}{\overline{M}_V^4} + \mathcal{O}(m_\pi^6) \right]^2, \\ \Gamma_{\sigma \rightarrow \pi\pi} &= \frac{3M_\sigma^3 \rho_{\sigma\pi\pi}}{16\pi F_\pi^4} c_d^2 \left[ 1 + \epsilon_S \frac{m_\pi^2}{\overline{M}_S^2} + \tilde{\epsilon}_S \frac{m_\pi^4}{\overline{M}_S^4} + \mathcal{O}(m_\pi^6) \right]^2.\end{aligned}\quad (40)$$

A way out to estimate these chiral corrections is the phenomenological lagrangian of Eq.(1):

$$\begin{aligned}\tilde{\epsilon}_V &= \epsilon_V \left[ \frac{8c_m(c_d - c_m)}{F^2} \frac{\overline{M}_V^2}{\overline{M}_S^2} + 4e_m^V \right], \\ \tilde{\epsilon}_S &= \frac{16c_m^2(c_d - c_m)}{c_d F^2} + \frac{8(c_m - c_d)e_m^S}{c_d},\end{aligned}\quad (41)$$

with  $\epsilon_V = 4\sqrt{2}f_\chi/\lambda_{V\pi\pi}$  and the mass splitting operators (11) also taken into account. The full unexpanded expression for the widths can be found in Ref.[2]. Using the inputs of former sections, one obtains

$$\begin{aligned}\tilde{\epsilon}_V &= -0.0 \pm 0.3 \text{ (for } n=5), & \tilde{\epsilon}_V &= -2.8 \pm 1.7 \text{ (for } n=3), \\ \tilde{\epsilon}_S &= -30 \pm 40.\end{aligned}\quad (42)$$

Gathering all the different contributions together, it is now possible to get the NNLO correction  $\gamma_R$  to the ratio  $\Gamma_R/M_R^3$ :

$$\begin{aligned}\gamma_V &= 2\tilde{\epsilon}_V - 2(n-3)\tilde{e}_m^V - 4\delta F_{(4)} \frac{\overline{M}_V^4}{\overline{M}_S^4} \\ &+ \epsilon_V^2 - 4(n-3)e_m^V \epsilon_V - 8\epsilon_V \delta F_{(2)} \frac{\overline{M}_V^2}{\overline{M}_S^2} - 12\epsilon_V + 2(n^2 - 8n + 15)e_m^V \\ &+ 10(\delta F_{(2)})^2 \frac{\overline{M}_V^4}{\overline{M}_S^4} + 12(n-5)e_m^V + 24\delta F_{(2)} \frac{\overline{M}_V^2}{\overline{M}_S^2} + 8(n-3)e_m^V \delta F_{(2)} \frac{\overline{M}_V^2}{\overline{M}_S^2} + 6, \\ \gamma_S &= 2\tilde{\epsilon}_S - 4\delta F_{(4)} \\ &+ \epsilon_S^2 - 4\epsilon_S - 8\epsilon_S \delta F_{(2)} + 8\delta F_{(2)} + 10(\delta F_{(2)})^2 - 8e_m^S - 2.\end{aligned}\quad (43)$$

Notice that the  $\mathcal{O}(m_\pi^4)$  corrections to  $F_\pi$  ( $\delta F_{(4)}$ ), the resonance masses ( $\tilde{e}_m^R$ ) and widths ( $\tilde{\epsilon}_R$ ) are presented in the first line of Eqs. (43)–(44). The results of Eqs. (43)–(44) are actually model dependent since there exist several other resonance operators that may contribute to the NNLO parameters  $\delta F_{(4)}$ ,  $\tilde{\epsilon}_V$ ,  $\tilde{\epsilon}_S$  [11]. Nevertheless, if one relies in the model of Eq. (1) the inputs considered in former sections lead to the chiral symmetry breaking terms

$$\begin{aligned}\gamma_V &= 30 \pm 10 \text{ (for } n=5), & \gamma_V &= 19 \pm 8 \text{ (for } n=3), \\ \gamma_S &= -50 \pm 70.\end{aligned}\quad (45)$$

A more detailed observation of the NNLO parameters (first line of Eqs. (43)–(44)) shows that the impact of the NNLO mass splitting term  $\tilde{e}_m^R$  is negligible. On the other hand,  $\delta F_{(4)}$  and  $\tilde{\epsilon}_S$  are responsible of most of the error in  $\gamma_S$ . In the vector case, the effect of the width splitting term  $\tilde{\epsilon}_V$  is not dominant, whereas the  $F_\pi$  NNLO correction  $\delta F_{(4)}$  is responsible of roughly the 75% of the total uncertainty in  $\gamma_V$ . The large errors we have in our input numbers for  $\delta F_{(4)}$  and  $\tilde{\epsilon}_S$  reassure us in the validity of our result, even if the uncertainty of the  $\gamma_R$  covers a rather conservative interval.

#### IV. LOW-ENERGY CONSTANT DETERMINATION AT $\mathcal{O}(p^6)$

##### A. The determination of $r_5, r_6$

The couplings  $r_5, r_6$  only depend on the chiral limit values of the resonance masses and decay widths [2] and, therefore, are the most reliably determined low-energy constants:

$$r_5 = \frac{32\pi F^6 \overline{\Gamma}_\sigma}{3\overline{M}_S^7} + \frac{36\pi F^6 \overline{\Gamma}_\rho}{\overline{M}_V^7}, \quad (46)$$

$$r_6 = \frac{12\pi F^6 \overline{\Gamma}_\rho}{\overline{M}_V^7}. \quad (47)$$

##### B. Extraction of $r_3, r_4$

In the case of the couplings  $r_3$  and  $r_4$ , all one needs are the resonance widths and masses in the chiral limit and the first  $m_\pi^2$  correction to the ratio  $\Gamma_R/M_R^5$ , this is,  $\beta_R$  [2]:

$$r_3 = \frac{64\pi F^6 \overline{\Gamma}_\sigma}{3\overline{M}_S^7} \left(1 + \frac{\beta_S}{2}\right) - \frac{768\pi F^6 \overline{\Gamma}_\rho}{\overline{M}_V^7} \left(1 + \frac{3\beta_V}{32}\right), \quad (48)$$

$$r_4 = \frac{192\pi F^6 \overline{\Gamma}_\rho}{\overline{M}_V^7} \left(1 + \frac{\beta_V}{8}\right). \quad (49)$$

The coupling  $r_4$  depends only on the vector resonance. Likewise, the scalar contribution is quite suppressed in  $r_3$  due to the large numerical coefficient in front of the vector term.

##### C. Extraction of $r_2$

Compared to the determination of  $r_3, \dots, r_6$ , the coupling  $r_2$  carries larger theoretical uncertainties since the NNLO parameters  $\gamma_S$  and  $\gamma_V$  enter into play. The prediction for  $r_2$  derived from partial-wave dispersion relations is given by [2]

$$r_2 = 2r_F + \frac{64\pi F^6 \overline{\Gamma}_\sigma}{\overline{M}_S^7} \left(1 + \frac{\beta_S}{3} + \frac{\gamma_S}{6}\right) + \frac{\pi F^6 \overline{\Gamma}_\rho}{\overline{M}_V^7} (7584 + 1248\beta_V + 144\gamma_V), \quad (50)$$

where  $r_F$  provides  $F_\pi$  at NNLO in  $m_\pi^2$  [4]. It is related to our previously defined  $\delta F_{(4)}$  through  $r_F = 2\ell_3\ell_4 + F^4\delta F_{(4)}/\overline{M}_S^4$ . The phenomenological lagrangian (1) produces the value of  $\delta F_{(4)}$  in Eq. (32), which in combination with  $2\ell_3\ell_4 = 32c_m^2 c_d(c_m - c_d)/\overline{M}_S^4$  [10] yields

$$r_F = -\frac{8c_d^2 c_m^2}{\overline{M}_S^4} + \frac{16e_m^S c_d c_m F^2}{\overline{M}_S^4}. \quad (51)$$

Substituting our former phenomenological inputs, one gets  $r_F = (-1 \pm 3) \cdot 10^{-4}$ .

##### D. Extraction of $r_1$ and $r_{S2}$

Within the framework of  $\pi\pi$  partial-wave sum-rules proposed in Refs. [1, 2], it is not possible to make any prediction for  $r_1$ , as they are based on once-subtracted dispersion relation and this coupling produces just a constant contribution to the  $\pi\pi$ -scattering amplitude.

The couplings  $r_1$  (from  $\pi\pi$ -scattering) and  $r_{S2}$  (from the  $\pi\pi$  scalar form-factor) must be extracted through alternative procedures. The values from Eqs. (8) and (9) were based on the lagrangian (1) but without accounting for the mass splitting effect. If the latter is included, our phenomenological model produces for  $r_1$  the new predictions

$$\begin{aligned} r_1^{n=5} &= -\frac{16c_dc_m(8c_d^2 - 17c_dc_m + 12c_m^2)}{\overline{M}_S^4} + \frac{32(c_d - c_m)^2 F^2}{\overline{M}_S^4} e_m^S \\ &\quad - \frac{16g_V^2 F^2}{\overline{M}_V^2} \left[ 1 + \epsilon_V + \frac{1}{4}\epsilon_V^2 - \frac{8c_dc_m}{F^2} \frac{\overline{M}_V^2}{\overline{M}_S^2} \right], \\ r_1^{n=3} &= -\frac{16c_dc_m(8c_d^2 - 17c_dc_m + 12c_m^2)}{\overline{M}_S^4} + \frac{32(c_d - c_m)^2 F^2}{\overline{M}_S^4} e_m^S \\ &\quad - \frac{16G_V^2 F^2}{\overline{M}_V^4} \left[ 1 + \epsilon_V - \frac{8c_dc_m}{F^2} \frac{\overline{M}_V^2}{\overline{M}_S^2} + 2e_m^V \right]. \end{aligned} \quad (52)$$

The detailed calculation is relegated to Appendix A 1.

Likewise, after taking into account the mass splitting in the scalar form-factor calculation (Appendix A 2),  $r_{S2}$  becomes

$$r_{S2} = \frac{8c_m(c_m - c_d)F^2}{\overline{M}_S^4} - \frac{32c_d^2 c_m^2}{\overline{M}_S^4} + \frac{16c_dc_m F^2}{\overline{M}_S^4} e_m^S. \quad (53)$$

### E. Saturation scale uncertainty

One last problem to face is the fact that the large- $N_C$  estimate of the LECs does not carry any renormalization scale dependence. However, it is possible to find a so called “saturation scale”  $\mu_s$  such that the physical LEC  $r_i^r(\mu)$  agrees numerically with  $r_i^{N_C \rightarrow \infty}$  for  $\mu = \mu_s$ .

Since the standard comparison scale is  $\mu_0 = 770$  MeV, the coupling  $r_i^r(\mu)$  must be run from  $\mu_s$  up to  $\mu_0$ . The possible difference between these two scales introduces an uncertainty in our determination. The way considered here to account for this lack of knowledge is to observe the variation for a wide range of  $\mu$ , which in other works is usually taken to be in the range 500–1000 MeV [3, 6, 11, 25]:

$$\begin{aligned} \Delta r_1^{\mu_s} &= 3 \times 10^{-4}, \quad \Delta r_2^{\mu_s} = 4 \times 10^{-4}, \quad \Delta r_3^{\mu_s} = 3 \times 10^{-4}, \\ \Delta r_4^{\mu_s} &= 0.05 \times 10^{-4}, \quad \Delta r_5^{\mu_s} = 0.5 \times 10^{-4}, \quad \Delta r_6^{\mu_s} = 0.05 \times 10^{-4}, \\ \Delta r_{S2}^{\mu_s} &= 1.5 \times 10^{-4}, \end{aligned} \quad (54)$$

where the running is completely fixed in  $\chi$ PT by the expressions given in Appendix B in terms of  $F$  and the  $\mathcal{O}(p^4)$  invariants  $\bar{\ell}_1 = -0.4 \pm 0.6$ ,  $\bar{\ell}_2 = 4.3 \pm 0.1$ ,  $\bar{\ell}_4 = 4.4 \pm 0.2$  [6] and  $\bar{\ell}_3 = 2.9 \pm 2.4$  [26].

Sometimes, it is argued that there must exist a common saturation scale for all the LECs, both  $\mathcal{O}(p^4)$  and  $\mathcal{O}(p^6)$ . This scale  $\mu_s$  is very often identified with the rho mass,  $\mu_s = M_\rho$ . However, explicit one-loop calculations with resonance lagrangians show that, though the loops typically produce logarithms of the form  $\ln \frac{M_R}{\mu}$  or  $\ln \frac{M_R}{M_R}$ , the combinations of them appearing for each LEC are not necessarily the same [27, 28, 29, 30]. Hence, although in general terms  $\mu_s \sim M_R$ , nothing ensures that the saturation scales must be exactly identical, nor that their value must be equal to  $M_\rho$  [30, 31]. Thus, the uncertainty of every LEC is computed and accounted separately in the present work.

### F. Summary of low-energy constants

The values for the low-energy constants  $r_i^r(\mu)$  and the corresponding errors are gathered in Table I. We choose the standard comparison scale  $\mu = 770$  MeV and include the uncertainty from the saturation scale estimated in the former section as an error.

Some remarks about the present predictions are in order. The LECs  $r_5$  and  $r_6$  are the most model-independent ones since they are determined by just the chiral limit of the resonances masses and widths (see Eqs.(46) and (47)).

	ND est. [32]	set A [3, 4]	set C (n=5)	set C (n=3)
$10^4 \cdot r_1^r$	$\pm 80$	-0.6	$-14 \pm 17 \pm 3$	$-20 \pm 17 \pm 3$
$10^4 \cdot r_2^r$	$\pm 40$	1.3	$22 \pm 16 \pm 4$	$7 \pm 10 \pm 4$
$10^4 \cdot r_3^r$	$\pm 20$	-1.7	$-3 \pm 1 \pm 3$	$-4 \pm 1 \pm 3$
$10^4 \cdot r_4^r$	$\pm 3$	-1.0	$-0.22 \pm 0.13 \pm 0.05$	$0.13 \pm 0.13 \pm 0.05$
$10^4 \cdot r_5^r$	$\pm 6$	1.1	$0.9 \pm 0.1 \pm 0.5$	$0.9 \pm 0.1 \pm 0.5$
$10^4 \cdot r_6^r$	$\pm 2$	0.3	$0.25 \pm 0.01 \pm 0.05$	$0.25 \pm 0.01 \pm 0.05$
$10^4 \cdot r_{S_2}^r$	$\pm 1$	-0.3	$1 \pm 4 \pm 1$	$1 \pm 4 \pm 1$

TABLE I: Different predictions for the  $\mathcal{O}(p^6)$  LECs  $r_i^r(\mu)$  for  $\mu = 770$  MeV: The first column presents the order of magnitude estimate based on naive dimensional analysis [32]; In the set A column we show former estimates from Refs. [3, 4]; in the last two columns, one can find the values for the present reanalysis, both in the Proca (n=5) and antisymmetric vector formalism (n=3). The first error derives from the inputs and the second from the uncertainty in the saturation scale.

Our determinations for  $r_5$  and  $r_6$  are consistent with those in Ref. [3]. The constants  $r_3$  and  $r_4$  depend on the  $\mathcal{O}(m_\pi^2)$  corrections  $\beta_R$  in the ratio  $\Gamma_R/M_R^5$ , which are still rather under control. All our predictions are in agreement within errors with those reported in Ref. [3] except for the small deviation found for  $r_4$ , mainly due to the slight discrepancies in the coupling  $f_\chi$  mentioned in previous sections. All this points out the little model dependence of our LEC estimate.

On the other hand,  $r_2$  is partly determined by the NNLO  $m_\pi^2$  corrections  $\gamma_R$  to the ratio  $\Gamma_R/M_R^3$ . This makes it one of the least controlled LECs. Roughly half of the total error of the LEC is due to the NNLO  $m_\pi^2$  corrections  $\delta F_{(4)}$  and  $\tilde{\epsilon}_S$  in  $\gamma_S$ . Thus, the uncertainty of  $r_2$  is found to be dominated by the scalar mass parameters, which is due in our case to our poor knowledge of the  $I = 1/2$  scalar mass.

In general, the value of all the couplings that can be determined by means of the partial wave dispersion relations ( $r_2, \dots, r_6$ ) [1, 2] is found to be dominated by the vector contributions. Since this sector is rather under control, this reassures us in the robustness of our calculation.

Finally, the couplings  $r_1$  and  $r_{S_2}$  need to be estimated through a phenomenological model. In the case of the lagrangian (1) the error of  $r_1$  is mainly due to our ignorance on the scalar resonances couplings  $c_d$  and  $c_m$ . This is directly originated in the large uncertainty of the  $a_0 \rightarrow \pi\eta$  decay width. The vector contribution is much smaller and essentially negligible. The error in  $r_{S_2}$  stems mainly from our poor knowledge of the  $a_0(980)$  width and the scalar mass splitting (which derives here from the large uncertainty of the  $I = 1/2$  scalar resonance).

In summary, we find in general a good agreement with former determinations and at the same time we are able to provide a reliable estimate of the error. This will help us to establish the relevance of the  $\mathcal{O}(p^6)$  LEC contributions to the scattering lengths in next section.

	Set A [3] ( $\times 10^{-3}$ )	Set C (n=5) ( $\times 10^{-3}$ )	Set C (n=3) ( $\times 10^{-3}$ )
$a_0^0$	1.2	$2.3 \pm 1.3 \pm 1.7$	$-0.3 \pm 1.0 \pm 1.7$
$b_0^0$	4.1	$4 \pm 1 \pm 4$	$2 \pm 1 \pm 4$
$10a_0^2$	-3.7	$-4 \pm 4 \pm 1$	$-4 \pm 4 \pm 1$
$10b_0^2$	14	$-13 \pm 8 \pm 2$	$-3 \pm 5 \pm 2$
$10a_1^1$	-0.3	$3.4 \pm 2.2 \pm 0.6$	$1.7 \pm 1.3 \pm 0.6$
$10b_1^1$	0.7	$2.6 \pm 0.5 \pm 1.5$	$3.7 \pm 0.6 \pm 1.5$

TABLE II: Contribution of the  $\mathcal{O}(p^6)$  LECs  $r_i^r(\mu)$  to the scattering lengths  $a_J^I$  and effective ranges  $b_J^I$  in the direct  $\chi$ PT approach [3] for  $\mu = 770$  MeV. The first column (set A) shows the results from Ref. [3], with their same inputs  $F_\pi = 93.2$  MeV,  $m_\pi = 139.47$  MeV. The last two columns show the predictions based on our phenomenological reanalysis (set C) for the Proca (n=5) and antisymmetric tensor formalisms (n=3). The first error derives from the inputs and the second one from the saturation scale uncertainty.

## V. SCATTERING LENGTHS

### A. Direct $\chi$ PT calculation

The  $\pi\pi$ -scattering lengths were first calculated in  $\chi$ PT up to  $\mathcal{O}(p^6)$  in Ref. [3]. At order  $m_\pi^6$ , the contribution from the  $r_i^r(\mu)$  LECs was found to be

$$\begin{aligned}
a_0^0|_{r_i} &= \frac{m_\pi^6}{32\pi F_\pi^6} [5r_1^r + 12r_2^r + 48r_3^r + 32r_4^r + 192r_5^r] , \\
b_0^0|_{r_i} &= \frac{m_\pi^6}{4\pi F_\pi^6} [r_2^r + 12r_3^r + 12r_4^r + 72r_5^r - 8r_6^r] , \\
a_0^2|_{r_i} &= \frac{m_\pi^6}{16\pi F_\pi^6} [r_1^r + 16r_4^r] , \\
b_0^2|_{r_i} &= \frac{m_\pi^6}{8\pi F_\pi^6} [-r_2^r + 24r_4^r - 16r_6^r] , \\
a_1^1|_{r_i} &= \frac{m_\pi^6}{24\pi F_\pi^6} [r_2^r + 8r_4^r + 16r_6^r] , \\
b_1^1|_{r_i} &= \frac{m_\pi^6}{6\pi F_\pi^6} [-r_3^r + 3r_4^r + 8r_6^r] .
\end{aligned} \tag{55}$$

In order to work with dimensionless quantities, we have multiplied the results in Ref. [3] for  $b_0^0$ ,  $b_0^2$ ,  $a_1^1$  by  $m_\pi^2$ , and for  $b_1^1$  by  $m_\pi^4$ . Our estimate of the contributions to the scattering lengths and effective ranges for the standard comparison scale  $\mu = 770$  MeV is given in Table II, both for the Proca (n=5) and the antisymmetric vector formalism (n=3). In order to have a clearer comparison of this quantities, the outcome from Ref. [3] is also provided. The first error in Table II comes from the phenomenological inputs and the second one from the saturation scale uncertainty in the  $\mathcal{O}(p^6)$  LECs, given in Eq.(54).

An important part of the subdominant quark mass corrections has been pinned down quite accurately through the comparison of the decays of the different resonances in the multiplet. However, there exist a series of new operators (e.g. the vector resonance operator  $\lambda_9^V$  in Ref. [11]) whose couplings cannot be extracted in an independent way. If present, they could enter in effective combinations  $g_V(m_\pi)^{\text{eff}}$  and  $c_d(m_\pi)^{\text{eff}}$ ; these would be what we would really determine and denote respectively as  $g_V$  and  $c_d$ , being used later instead of them in the LEC computation. A more detailed discussion is relegated to Appendix C. In any case, the study of these operators remains beyond the scope of this article.

### B. CGL dispersive method

In Ref. [6], Colangelo *et al.* combined the NNLO chiral perturbation theory computation of the scattering lengths [3] with a phenomenological dispersive representation. This allowed them to produce one of the most precise determina-

tions of the scattering lengths. These were expressed in terms of some dispersive integrals, the pion quadratic scalar radius  $\langle r^2 \rangle_S^\pi$ , the  $\mathcal{O}(p^4)$  coupling  $\ell_3$  and a set of  $\mathcal{O}(p^6)$  LECs ( $r_1, \dots, r_4, r_{S_2}$ ). These last contributions are actually the most poorly known and the aim of this paper. Following the work of Ref.[6], we extract the part of their scattering lengths that depends on the inputs  $r_i^r(\mu)$ :

$$\begin{aligned} a_0^0|_{r_i} &= \frac{7m_\pi^2}{32\pi F_\pi^2} C_0|_{r_i} = \frac{m_\pi^6}{32\pi F_\pi^6} [5r_1^r + 12r_2^r + 28r_3^r - 28r_4^r - 14r_{S_2}^r], \\ a_0^2|_{r_i} &= -\frac{m_\pi^2}{16\pi F_\pi^2} C_2|_{r_i} = \frac{m_\pi^6}{16\pi F_\pi^6} [r_1^r - 4r_3 + 4r_4 + 2r_{S_2}^r], \end{aligned} \quad (56)$$

where the  $C_j|_{r_i}$  can be extracted from the  $C_j$  provided in the Appendix C of Ref. [6]. The LEC  $r_{S_2}$  appears in this analysis [6] because the scalar radius  $\langle r^2 \rangle_S^\pi$  is incorporated as an experimental information in order to fix the  $\mathcal{O}(p^4)$  constant  $\ell_4$ . On the other hand, the couplings  $r_5$  and  $r_6$  disappear here with respect to the standard  $\chi$ PT analysis of Ref. [3], being their information encoded and replaced by the different dispersive integrals. It is easy to realize that the resonance contributions  $a_J^I|_{r_i}$  carry the same  $r_1$  and  $r_2$  dependence in both the direct  $\chi$ PT calculation in Eq. (55) [3] and the dispersive study in Eq. (56) [6]. The comparison of this reanalysis and the predictions from Ref. [6], is shown in Table III for the standard reference scale  $\mu = 770$  MeV.

	Total: Ref. [6] ( $\times 10^{-3}$ )	$a_J^I _{r_i}$ [6] ( $\times 10^{-3}$ )	Set C (n=5) ( $\times 10^{-3}$ )	Set C (n=3) ( $\times 10^{-3}$ )
$a_0^0$	$220 \pm 5$	$0.0 \pm 1.0$	$1.0 \pm 1.5 \pm 1.0$	$-1.6 \pm 1.5 \pm 1.0$
$10a_0^2$	$-444 \pm 10$	$0.4 \pm 2.0$	$0 \pm 4 \pm 2$	$0 \pm 4 \pm 2$

TABLE III: The first and second columns show, respectively, the total scattering lengths and the  $r_i$  contribution to them in the dispersive method from Colangelo *et al.* [6], where the authors used the  $r_i$  in Eq. (2.3),  $F_\pi = 92.4$  MeV and  $m_\pi = 139.57$  MeV. The last two columns show the reanalyzed quantities (set C) for the Proca (n=5) and antisymmetric formalisms (n=3) for the usual scale  $\mu = 770$  MeV. There, the first error derives from the inputs and the second one from the saturation scale uncertainty.

We find that the largest contributions to the  $a_0^0$  and  $a_0^2$  errors are produced in similar terms by  $r_1$ ,  $r_2$ ,  $r_3$  and  $r_{S_2}$ . On the other hand, the impact of  $r_4$  on both the value and uncertainty of the scattering lengths results negligible.

## VI. SUMMARY AND CONCLUSION

This article concludes the previous work from Ref. [2]. The analysis of the uncertainties of the  $\mathcal{O}(p^6)$  LECs contributing to the  $\pi\pi$  scattering lengths [3, 6] is completed here. Former estimates based on the large- $N_C$  limit and resonance saturation have been revised. Nevertheless, a series of uncertainties escape to the control of the analysis carried in the present article. Most of the computations are rather model independent as they are based on general resonance properties; however, at some points we had to rely on phenomenological lagrangians [3, 10, 15].

All this allowed the estimate of the  $\mathcal{O}(p^6)$  LECs. For the standard comparison scale  $\mu = 770$  MeV, one obtains after combining the results from Proca and antisymmetric formalism the values

$$\begin{aligned} r_1^r &= (-17 \pm 20) \times 10^{-4}, & r_2^r &= (17 \pm 21) \times 10^{-4}, & r_3^r &= (-4 \pm 4) \times 10^{-4}, \\ r_4^r &= (0.0 \pm 0.3) \times 10^{-4}, & r_5^r &= (0.9 \pm 0.5) \times 10^{-4}, & r_6^r &= (0.25 \pm 0.05) \times 10^{-4}, \\ r_{S_2}^r &= (1 \pm 4) \times 10^{-4}. \end{aligned} \quad (57)$$

The combination of the results from the Proca (n=5) and antisymmetric formalism (n=3) yield for the dispersive method [6] the final resonance contribution,

$$10^3 a_0^0|_{r_i} = 0 \pm 3, \quad 10^4 a_0^2|_{r_i} = 0 \pm 5. \quad (58)$$

This is in perfect agreement with the former estimate [6], although the detailed analysis of the phenomenological inputs casts a slightly larger uncertainty.

Following the analysis of global uncertainties of the scattering lengths in Colangelo *et al.*'s [6], we have verified that the global uncertainties for  $a_0^0$  and  $a_0^2$  are not largely modified. Thus, the replacement of the values from Eq. (58) in the total scattering lengths leads to the updated predictions

$$a_0^0 = 0.220 \pm 0.005, \quad 10a_0^2 = -0.444 \pm 0.011, \quad (59)$$

where the total uncertainties and central values are essentially unchanged with respect to the previous determinations  $a_0^0 = 0.220 \pm 0.005$  and  $10a_0^2 = -0.444 \pm 0.010$  [6].

This work provides a reliable and solid estimation of this part of the  $\mathcal{O}(p^6)$  calculation. It sets clear limits to the size of the  $\mathcal{O}(p^6)$  LEC contributions, slightly conservative in some occasions. A special attention has been put on the quantification of errors and their precise source. It reassures us in the reliability and precision of the current scattering length determination based on the dispersive approach of Ref. [6]. Nonetheless, although the global uncertainties of the scattering lengths are barely affected by our new predictions of the  $\mathcal{O}(p^6)$  low energy constants, the total errors have almost reached those from  $a_J^I|_{r_i}$ . This points out the difficulty of further improvements in the accuracy of the  $\pi\pi$  scattering lengths unless the  $\mathcal{O}(p^6)$  LEC uncertainties are conveniently reduced.

### Acknowledgements

This work has been supported in part by CICYT-FEDER-FPA2008-01430, SGR2005-00916, the Spanish Consolider-Ingenio 2010 Program CPAN (CSD2007-00042), the Juan de la Cierva program, the EU Contract No. MRTN-CT-2006-035482 (FLAVIANet) and National Nature Science Foundation of China under grant number 10875001, 10575002 and 10721063. We are in debt with R. Escribano for his comments on how to improve the treatment of the  $\eta$  meson in the  $a_0 \rightarrow \eta\pi$  decay.

## APPENDIX A: FULL CALCULATION BASED ON THE RESONANCE LAGRANGIAN

### 1. Extraction of $r_1$ : $\pi\pi$ scattering

The resonance lagrangian from Eq. (1) [3, 10] has been used to calculate the  $\pi\pi$ -scattering amplitude at large  $N_C$ . The vector resonances were described there by means of Proca four-vector fields  $\hat{V}_\mu$ . In addition, we also took into account the vector and scalar mass splittings. The amplitude  $\pi^+(p_1)\pi^-(p_2) \rightarrow \pi^0(p_3)\pi^0(p_4)$  results then

$$A(s, t, u)^{\text{Proca}} = \frac{s - m_\pi^2}{F_\pi^2} + \frac{2}{F_\pi^4(M_\sigma^2 - s)} [c_d s - 2c_d m_\pi^2 + 2c_m m^2]^2 \quad (\text{A1})$$

$$+ \frac{u - s}{F_\pi^4(t - M_\rho^2)} [g_V t + 4\sqrt{2}f_\chi m^2]^2 + \frac{t - s}{F_\pi^4(u - M_\rho^2)} [g_V u + 4\sqrt{2}f_\chi m^2]^2.$$

with  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ ,  $u = (p_1 - p_4)^2$ . We denote by  $\rho$  and  $\sigma$  respectively the  $I = 1$  vector and the  $I = 0$  scalar with  $\bar{u}u + \bar{d}d$  content.  $F_\pi \simeq 92.4$  MeV is the pion decay constants, which deviates from its chiral limit  $F$  in the way prescribed in Eq. (31) due to the scalar resonance tadpole in the  $c_m$  operator of the resonance chiral theory lagrangian (1) [2, 13]. The same happens with the physical pion mass  $m_\pi^2$  and its value at leading order in the chiral expansion,  $m^2$ , which are related through

$$m^2 \equiv 2B_0 m_{u/d} = m_\pi^2 + \frac{8c_m(c_d - c_m)}{f^2} \frac{m_\pi^4}{M_S^2} + \mathcal{O}(m_\pi^6), \quad (\text{A2})$$

with  $m_{u/d}$  the  $u/d$  quark mass in the isospin limit.

Alternatively, if one employs the antisymmetric tensor formalism to describe the vector resonances [10, 11, 26] the scattering takes the form

$$A(s, t, u)^{\text{Antis.}} = \frac{s - m_\pi^2}{F_\pi^2} + \frac{2}{F_\pi^4(M_\sigma^2 - s)} [c_d s - 2c_d m_\pi^2 + 2c_m m^2]^2$$

$$+ \frac{u - s}{F_\pi^4(t - M_\rho^2)} t \left[ G_V + 4\sqrt{2}f_\chi^{n=3} m^2 / M_\rho^2 \right]^2 + \frac{t - s}{F_\pi^4(u - M_\rho^2)} u \left[ G_V + 4\sqrt{2}f_\chi^{n=3} m^2 / M_\rho^2 \right]^2, \quad (\text{A3})$$

where the meaning of  $G_V$  and  $f_\chi^{n=3}$  will be discussed later in detail in Appendix C.



The resonance expressions can be now compared to the  $\chi$ PT amplitude [3],

$$\begin{aligned}
A(s, t, u)^{\chi\text{PT}} = & \frac{s - m_\pi^2}{F^2} + \frac{m_\pi^4}{F^4} (8\ell_1 + 2\ell_3) - \frac{8m_\pi^2 s}{F^4} \ell_1 + \frac{s^2}{F^4} (2\ell_1 + \frac{\ell_2}{2}) \\
& + \frac{(t - u)^2}{2F^4} \ell_2 - \frac{8m_\pi^6}{F^6} \ell_3^2 + \frac{m_\pi^6}{F^6} (r_1 + 2r_F) + \frac{m_\pi^4 s}{F^6} (r_2 - 2r_F) \\
& + \frac{m_\pi^2 s^2}{F^6} r_3 + \frac{m_\pi^2 (t - u)^2}{F^6} r_4 + \frac{s^3}{F^6} r_5 + \frac{s(t - u)^2}{F^6} r_6.
\end{aligned} \tag{A4}$$

We have preferred to express everything in terms of  $F$  rather than  $F_\pi$  in order to make the chiral matching more transparent [1, 2].

The LECs are extracted by matching  $A(s, t, u)^{\chi\text{PT}}$  and the chiral expansion of the resonance amplitude  $A(s, t, u)^{\text{Res}}$  in Eqs. (A1) and (A3). Since we are interested in  $r_1$ ,  $s = t = 0$  and  $u = 4m_\pi^2$  turn out to be the most convenient choice of momenta. The matching of the  $m_\pi^6$  term yields for the Proca formalism (n=5)

$$\begin{aligned}
r_1 + 2r_F + 16r_4 - 8\ell_3^2 = & -\frac{128f_\chi^2 F^2}{\overline{M}_V^2} - 64 \frac{c_m(2c_d^3 - 2c_d^2 c_m - c_d c_m^2 + 2c_m^3)}{\overline{M}_S^4} \\
& - \frac{32e_m^S F^2}{\overline{M}_S^4} (c_m^2 - c_d c_m + c_d^2),
\end{aligned} \tag{A5}$$

with the term  $-128f_\chi^2 F^2 / \overline{M}_V^2$  in the right-hand side absent in the antisymmetric tensor formalism (n=3). However, for the case of our phenomenological lagrangian, its numerical value is absolutely negligible. Thus, the vector contribution comes mainly from the LEC  $r_4 = \frac{\lambda_{V\pi\pi}^2 F^2}{M_V^{4-n}} \left(1 + \epsilon_V - (n-5)e_m^V - \frac{2\overline{M}_V^2}{\overline{M}_S} \delta F_{(2)}\right)$ , which is extracted from the large- $N_C$  partial wave analysis in a very reliable way [2]. Likewise, although  $r_F$  is obtained by means of the resonance lagrangian (1), the large errors we found for  $\delta F_{(4)}$  in Eq. (33) make its estimate slightly conservative. For  $\ell_3$  we use,

$$\ell_3 = 4 \frac{c_m(c_m - c_d)}{\overline{M}_S^2}, \tag{A6}$$

where we converted the  $SU(3)$  large- $N_C$  estimate from Ref. [10] into  $SU(2)$  LECs [33]. Possible pseudo-scalar resonances have been neglected. Putting all this information together, one obtains the value for  $r_1$  reported in the text in Eq. (52).

## 2. Extraction of $r_{S2}$ : $\pi\pi$ scalar form factor

We proceed now to the calculation of the pion scalar form-factor by means of the resonance chiral theory lagrangian (1) [3, 10, 11]. We repeat the calculation in detail, including also the contributions from the scalar tadpole operator  $c_m$  [12, 13], and obtain

$$\mathcal{F}_S(s) = 2B_0 \left[ 1 + \frac{8c_m(c_m - c_d)m_\pi^2}{\overline{M}_S^2 F^2} + \frac{4c_m(c_d s - 2c_d m_\pi^2 + 2c_m m^2)}{F_\pi^2 (M_\sigma^2 - s)} + \mathcal{O}\left(\frac{m_\pi^4}{M_S^4}\right) \right]. \tag{A7}$$

Matching the resonance description in Eq.(A7) and the  $\chi$ PT result for the  $\pi\pi$  scalar form-factor [4] leads to the LECs

$$r_{S3} = 4F^2 \frac{c_m c_d}{\overline{M}_S^4}, \tag{A8}$$

$$r_{S2} - 4\ell_3 \ell_4 = \frac{8c_m(c_m - c_d)F^2}{\overline{M}_S^4} + \frac{32c_m^2 c_d (c_d - 2c_m)}{\overline{M}_S^2} + \frac{16c_d c_m F^2}{\overline{M}_S^4} e_m^S, \tag{A9}$$

with  $r_{S3}$  consistent with the value from Ref. [4]. Substituting the large- $N_C$  value of  $\ell_3$  from Eq. (A6) and  $\ell_4 = 4c_m c_d / \overline{M}_S^2$  leads to the prediction for  $r_{S2}$  shown in the text in Eq.(53).

## APPENDIX B: $\mathcal{O}(p^6)$ LOW-ENERGY CONSTANT RUNNING

The variation of the  $r_i^r(\mu)$  with  $\mu$  is given by the general pattern [3, 4]

$$r_i^r(\mu_1) - r_i^r(\mu_2) = \frac{K_i^{(L)}}{(16\pi^2)^2} \ln \frac{\mu_1}{\mu_2} + \frac{K_i^{(2)}}{(16\pi^2)^2} \left[ \ln^2 \frac{\mu_1}{m_\pi} - \ln^2 \frac{\mu_2}{m_\pi} \right], \quad (\text{B1})$$

with the scale invariants

$$\begin{aligned} K_1^{(L)} &= \frac{193}{27} + \frac{104}{9} \bar{\ell}_1 + \frac{112}{9} \bar{\ell}_2 - 6 \bar{\ell}_3 + 2 \bar{\ell}_4, & K_1^{(2)} &= -20, \\ K_2^{(L)} &= -\frac{556}{27} - \frac{68}{3} \bar{\ell}_1 - \frac{248}{9} \bar{\ell}_2 + 7 \bar{\ell}_3 - 2 \bar{\ell}_4, & K_2^{(2)} &= \frac{407}{9}, \\ K_3^{(L)} &= \frac{755}{108} + \frac{100}{9} \bar{\ell}_1 + \frac{44}{3} \bar{\ell}_2, & K_3^{(2)} &= -\frac{232}{9}, \\ K_4^{(L)} &= -\frac{1}{108} - \frac{2}{9} \bar{\ell}_1 - \frac{4}{9} \bar{\ell}_2, & K_4^{(2)} &= \frac{2}{3}, \\ K_5^{(L)} &= -\frac{29}{432} - \frac{7}{4} \bar{\ell}_1 - \frac{107}{36} \bar{\ell}_2, & K_5^{(2)} &= \frac{85}{18}, \\ K_6^{(L)} &= -\frac{79}{432} - \frac{5}{36} \bar{\ell}_1 - \frac{25}{36} \bar{\ell}_2, & K_6^{(2)} &= \frac{5}{6}, \\ K_{S_2}^{(L)} &= \frac{148}{27} + \frac{62}{9} \bar{\ell}_1 + 2 \bar{\ell}_3 + 2 \bar{\ell}_4, & K_{S_2}^{(2)} &= -\frac{166}{9}. \end{aligned} \quad (\text{B2})$$

## APPENDIX C: CHIRAL CORRECTIONS TO THE DECAY WIDTH

In the article, the splitting in the vector decay widths due to quark mass effects was parametrized at NLO by one single parameter  $\epsilon_V$  (or equivalently  $f_\chi$ ). This was true for the Proca resonance lagrangian in Eq. (1). However, in the case of a more general hadronic action, one single parameter  $\epsilon_V$  does not seem enough to describe the NLO chiral corrections to the decay widths. In the Proca field realization, in addition to the operator  $f_\chi \langle \hat{V}_\mu [u^\mu, \chi_-] \rangle$ , there could exist higher derivative operators such as  $\langle \hat{V}_{\mu\nu} \{ \chi_+, u^\mu u^\nu \} \rangle$ , which also contribute to the chiral corrections to the decay widths at NLO in  $m_\pi^2$ . Nonetheless, we will focus our digression about higher terms in the lagrangian in the antisymmetric tensor formalism, where the possible operators related to the NLO quark mass corrections to the vector width have been already constructed in Ref. [11]:

$$i\lambda_8^V \langle V_{\mu\nu} \{ \chi_+, u^\mu u^\nu \} \rangle + i\lambda_9^V \langle V_{\mu\nu} u^\mu \chi_+ u^\nu \rangle + \lambda_{10}^V \langle V_{\mu\nu} [u^\mu, \nabla^\nu \chi_-] \rangle. \quad (\text{C1})$$

However, not all these three couplings are observable in the partial decay widths we analyzed in the text, which now become

$$\Gamma_{V \rightarrow \phi_1 \phi_2} = C_{V12} \frac{M_V^3 \rho_{V12}^3}{48 \pi F_1^2 F_2^2} G_V^{\text{eff}^2} \left[ 1 + \frac{2\sqrt{2} (2\lambda_8^V + \lambda_{10}^V)}{G_V^{\text{eff}}} (m_1^2 + m_2^2) + \mathcal{O}(m_P^4) \right]^2, \quad (\text{C2})$$

with the corresponding Clebsch-Gordan  $C_{V12}$  for the channels under consideration,  $C_{\rho\pi\pi} = 1$ ,  $C_{K^*K\pi} = 3/4$  and  $C_{\phi K \bar{K}} = 1$ . Instead of the  $m_\pi$ -independent coupling  $G_V$ , which we had before in Eq. (17), one has now the effective combination

$$G_V^{\text{eff}} = G_V + 2\sqrt{2} (\lambda_9^V - 2\lambda_8^V) m_\pi^2. \quad (\text{C3})$$

The combination of resonance couplings  $(2\lambda_8^V + \lambda_{10}^V)$  rules the splitting of our studied decays and determines the chiral symmetry breaking parameter  $f_\chi^{\text{n}=3} = (2\lambda_8^V + \lambda_{10}^V) \bar{M}_V^2$ .

Hence, the  $\rho \rightarrow \pi\pi$ ,  $K^* \rightarrow K\pi$  and  $\phi \rightarrow K\bar{K}$  partial decay widths only allow the determination of  $G_V^{\text{eff}}$ , not its chiral limit value  $G_V$ . A similar situation would happen with the scalar sector and the Proca field description for vectors, where the presence of higher operators could make us observe in our analysis some effective combinations  $c_d^{\text{eff}}$  and  $g_V^{\text{eff}}$ , instead of the quark mass independent couplings  $c_d$  and  $g_V$ . In any case, the difference is assumed to be small in the present work and its further study remains beyond the scope of this article.

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